MTH245 Unit 3 Module 1 RANDOMNESS and PROBABILITY

Probability –the likelihood that a random event will occur. In other words, how often an event will occur if an experiment is repeated a large number of times. If we flip a coin 1000 times, we would expect about half of the tosses to be heads and about half to be tails. But if we flip a coin twice we might observe exactly one head and one tail but would not be surprised to get two heads or two tails. When randomness is involved any possible event can occur, but some things are more likely to happen than others. Probability assigns a value p% (between 0 % and 100%) to the likelihood of an event occurring, if an experiment is repeated a large number of times we would expect to observe the event occurring p% of the time.

This will be a very brief and practical introduction to probability. Many of the traditional topics will be left for another course.

Events and Sample Space

When we were working with systems of equations and inequalities, we had to define variables before we could come up with objective functions and constraints. With probability we need to define **events** and **sample spaces** of an **experiment**. If these are not clearly defined it is remarkably easy to go completely wrong.

Example: Roll a single 6-sided die once ("the experiment")
What are the possible observations? These are called *simple outcomes* or *simple events*.
The set of all possible *simple outcomes* is called **the *sample space S***.

For this experiment, the sample space is the set: 1, 2, 3, 4, 5, 6
These are all of the possible outcomes of rolling a die.

An **event** is a subset of the sample space (including the empty set and *S* itself). The event is the specific outcome we are interested in.

Example : Consider the die rolling experiment. What outcomes of the sample space would be part of the following events?

E_1: Die comes up even: 2, 4, 6

E_2: Die comes up less than three: 1, 2, 3

E_3: Die comes up 1: 1

Again, the **sample space** is a list of all possible outcomes of an experiment. An **event** is the particular outcome we are interested in, and will be a subset of the sample space. If you start to get confused in a probability problem, back up and consider what the sample space of the experiment is. It really helps to clarify a problem.

Defining Sample Spaces
For some experiments there may be different sample spaces depending on what type of events we are interested in. Again, it is very important to think carefully about sample space before getting too far into a problem!

Example: Consider the experiment of recording the gender composition of two-child families.

A. What is an appropriate sample space if we want to record the gender of each child in age-order in the family? BB, BG, GB, GG
Are these outcomes equally likely? Yes, assuming that the boys are just as likely as girls. (Actually, the birth rate for boys is a smidge higher than that of girls)

B. What is an appropriate sample space if we are only interested in the number of boys in the family? 0, 1, 2
Are these events equally likely? No, there are two ways for 1 boy to occur: BG and GB so 1 boy is a more likely event than 0 boys or 2 boys.

Example: Give sample spaces for the following experiments

A. Flip a single coin once: H, T

B. Flip a coin twice and observe the outcomes in order: HH, HT, TH, TT

C. Flip two coins and observe the number of heads that appear: 0, 1, 2

A common random experiment in probability is rolling two or more fair 6-sided dice. All of the possible simple outcomes for rolling two such dice are shown below. This is the **sample space** for this experiment, since any question about the experiment can be answered by considering this sample space.

Sample Space for the Experiment "Rolling Two Dice"

Second Die First Die	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Write the **event** (subset of the sample space) corresponding to

E_1: A sum of 6 is rolled: (1, 5), (2, 4), (3, 3), (4, 2), (5, 1)

E_2. A double is rolled: (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)

E_3. The score on the first die is less than that on the second die: (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)

Having the sample space written out makes it easy to list out the ways specific events can occur. You may want to refer back to this chart for some future problems.

Now that we understand events and sample spaces, we can introduce a definition for probability. Note, this definition is ONLY for equally likely outcomes!

Probability of an Event – the measure of the degree of certainty of an event occurring. The probability of an event, E is defined by

$$P(E) = \frac{\text{number of ways for E to happen}}{\text{total number of } \textbf{equally likely} \text{ outcomes}}$$

0<=P(E)<= 1   (or 0%<=P(E)<=100%)

If P(E) = 0, then an event is impossible, If P(E) = 1, then an event is a sure thing.

If P(E) = 0.7 or 70%, then it is "pretty likely" to happen.

Example: Find the **probabilities** of the three events E_1, E_2, and E_3 when rolling two dice.

E_1: $P(sum\ is\ 6) = {}^{5}/_{36}$.

There are a total of 36 equally likely outcomes when rolling 2 dice, and 5 of them result in a sum of 6.

E_2: $P(double) = {}^{6}/_{36}$.

There are a total of 36 equally likely outcomes when rolling 2 dice, and 6 of them result in a double. Note, this is the only math class where it is OK not to reduce a fraction! In this situation it is easier to compare the probabilities if they all share a common denominator. (If you REALLY want to reduce the fraction, go ahead.)

E_3: $P(score\ on\ first < score\ on\ 2nd) = {}^{15}/_{36}$

There are a total of 36 equally likely outcomes when rolling 2 dice, and 15 of them result in the score on the first being less than the score on the second.

Example: Roll **a single** fair die; find the following probabilities:
P(1), P(4), P(1 or 4), P(number is less than 3) P(even) P(even or < 3)

First, list the sample space S = {1, 2, 3, 4, 5, 6}

Are these equally likely? Yes, as the die is fair each side is just as likely to land "up" as any other side.

$P(1) = {}^1\!/_6$	$P(4) = {}^1\!/_6$	$P(1\ or\ 4) = {}^2\!/_6$
$P(number\ is\ < 3) = {}^2\!/_6$	$P(even) = {}^3\!/_6$	$P(even\ or\ < 3) = {}^4\!/_6$
number < 3: 1, 2	*even:* 2, 4, 6	*even or < 3:* 2, 4, 6, 1

One more: $P(1\ or\ 2\ or\ 3\ or\ 4\ or\ 5\ or\ 6) = {}^6\!/_6 = 1$

Disjoint

Events are **disjoint** if you have defined them in such a way that there is no overlap. When rolling a single die, the event of "number < 3" overlaps the event of "even" (the number 2 is both < 3 and even), while the events of "1" and "4" or "even" and "odd" are disjoint.

Disjoint (no overlap)	Not disjoint (have overlap)
flip a coin: "head" and "tail" a baby is born: "male" and "female" student: "A" and "B" and "C" and "D" and "F" Draw a card: "Heart" and "Club"	flip a coin twice: "2 heads" and "at least 1 head" a baby is born: "brown eyes" and "male" student: "passes" and "needs to retake the class" Draw a card: "Ace" and "Heart"

Probability Rule #1: If two events E_1 and E_2 are **<u>disjoint</u>** (non-overlapping), then

$$P(E_1\ \underline{\textbf{or}}\ E_2) = P(E_1) + P(E_2)$$

(I remember this one as "you can add ors if there is no overlap")

A corollary to this rule is that:
IF events ARE disjoint, then the sum of the probabilities of **all** possible outcomes is equal to 1.

From our example of rolling a single die:

P(even) + P(odd) = 1 P(1) + P(2)+ P(3) +P(4) + P(5) + P(6) = 1

(Because something has to happen.)

Independence

Example: Roll **<u>two dice</u>** (recall the sample space). Find the following probabilities. (First identify the event as a set!)

A. $P(First\ roll\ is\ 2\ \boldsymbol{OR}\ second\ is\ 5) = {}^{11}/_{36}$

There are 6 ways the first roll is a 2 and 6 ways the second roll is a 5, but we cannot double count the event (2, 5)!

How does that compare to:

B. $P(first\ roll\ is\ 2\ \boldsymbol{AND}\ second\ is\ 5) = {}^{1}/_{36}$

There is only one way for this to occur: (2, 5)

"OR" and "AND" are two very different words and you will need to be careful. Let's talk about **independence**. Two events are independent if the outcome of one does not affect the likelihood of the other occurring. The outcome of a roll of a die has no effect on the outcome if I roll it a second time. Each flip of a coin is independent of the next flip.

BUT if I start talking about more complicated events I have to be careful. If I roll two dice, is the outcome "the sum of the dice is three" independent of the event "one of the die is even"?

Probability Rule #2: If two events E_1 and E_2 are **independent** (occurrence of the first doesn't affect the second), then

$$P(E_1\ and\ E_2) = P(E_1) \cdot P(E_2)$$

I remember this one as "you can multiply ands if independent"

Example: Roll two dice, find:

$P(first\ roll\ is\ even\ \boldsymbol{AND}\ second\ is\ even)$

$= P(first\ roll\ is\ even) \cdot P(second\ roll\ is\ even)$

$= {}^{3}/_{6} \cdot {}^{3}/_{6} = {}^{9}/_{36}$

Take a look back at the sample space for rolling two dice, how many have both the first roll and the second roll even numbers?

Complementary Events:

Recall that IF events ARE disjoint, then the sum of the probabilities of **all** possible outcomes is equal to 1. An event will either occur or it will not occur, correct?

P(event occurs) + P(event does not occur) = 1.

So P(event occurs) = 1 – P(event does not occur)

This can be a real timesaver.

Roll a single die, $P(1) = {}^{1}/_{6}, P(roll\ is\ greater\ than\ 1) = 1 - {}^{1}/_{6} = {}^{5}/_{6}$

Example Experiment: Observe gender distribution of children in 3-child families.

Find the following probabilities: P(all girls), P(no boys), P(at least one boy), P(two girls), P(at most 2 girls)

First, list the sample space:

BBB, BBG, BGB, GBB, GGB, GBG, BGG, GGG

P(all girls)= P(GGG) = 1/8   P(no boys)=P(GGG)=1/8
P(at least one boy)= 1 - P(no boys) = 1 - 1/8 = 7/8
P(two girls) = P(GGB or GBG or BGG) = 3/8
P(at most 2 girls) = 1 – P(more than 2 girls) = 1 – P(GGG) = 1- 1/8 = 7/8

Example (Roulette)
In roulette, a wheel with 38 numbers is spun. Of these, 18 are red, and 18 are black. The other two numbers, which are green, are 0 and 00. The probability when the wheel is spun that it lands on any particular number is 1/38.
If you spin two times, what is the probability that the wheel lands on a black number at least one time?

First consider the sample space: BB, B~~B~~, ~~B~~B, ~~BB~~ so in two spins can get either: 0 black, 1 black or 2 black. The event "at least one black" is equivalent to "1 black or 2 black" and the complementary event is "0 black".

P(at least 1 black) = 1 – P(0 black) = 1-P(~~BB~~)= 1 – P(1st pin not black and 2nd spin not black) = 1 – P(1st pin not black) * P(2nd spin not black) =1 – $(20/38)^2$ *remember that the spins are independent!*
This requires fewer calculations than: P(at least 1 black) = P(1st spin black and 2nd spin not black) + P(1st spin not black and 2nd spin black) + P(1st spin black and 2nd spin black) = P(B~~B~~) + P(~~B~~B) + P(BB)

Example (Genetics)
In a study to determine frequency and dependence of color blindness relative to males and females, 1000 people were chosen at random and the following numbers were observed:

	FEMALE (F)	MALE (M)	Totals
Color blind (*C*)	2	24	26
Normal (*N*)	518	456	974
Totals	520	480	1000

A person is randomly selected from the study. Find: P(male), P(color blind), P(male **and** color blind)

P(male)=480/1000   P(color blind)= 26/1000   P(male **and** color blind)=24/1000

Are gender and color blindness independent? Recall our definition of independence: If two events E_1 and E_2 are **independent** then: $P(E_1 \text{ } and \text{ } E_2) = P(E_1) \cdot P(E_2)$. We can turn this around: if $P(E_1 \text{ } and \text{ } E_2) \neq P(E_1) \cdot P(E_2)$ then E_1 and E_2 are **dependent.**
Does P(male **and** color blind)= P(male) P(color blind)? No,.024 $\neq$.48*.026. So the likelihood that someone is color blind is affected by the person's gender.

One more: What is the probability that a colorblind person is male?

For this situation we already know the person is colorblind so we only need to look at the first row of data in our table:

	FEMALE (F)	MALE (M)	Totals
Color blind (*C*)	2	24	26
Normal (*N*)	518	456	974
Totals	520	480	1000

$$P(a\ colorblind\ person\ is\ male) = \frac{number\ of\ males\ that\ are\ color\ blind}{number\ color\ blind} = \frac{24}{26}$$

Note: The *boys/girls in 3-child family* example and the *color blindness* example show two different types of probability.

1. **Theoretical probability** (boys/girls)
2. **Empirical probability** (color blindness)

Empirical probability will be discussed in the next module.