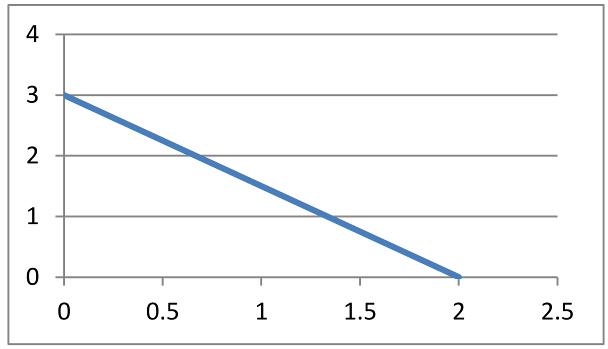
MTH245 Unit 2 Module 3 Linear Functions as Families and Systems

Standard Form of a Linear Equation

You are familiar with the Slope Intercept form of a line: $f(x) = mx + b$. Another form, that may be less familiar, is the Standard Form of a Line**: $Ax + By = C$**. Graphing the Standard Form is easily done by finding the x- and y-intercepts and then connecting the two points.

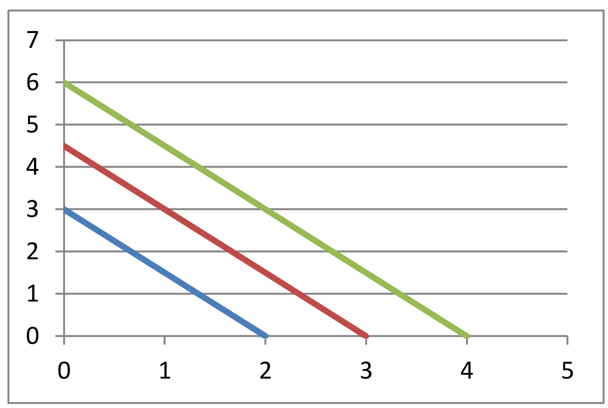


To graph $3x + 2y = 6$, let $x = 0$ and our equation becomes $2y = 6$, which gives us $y = 3$ and the y-intercept (0, 3). Next let $y = 0$ and the equation becomes $3x = 6$, giving us $x = 2$ and the x-intercept (2, 0). Plotting and connecting the two points gives us the graph of the line.

Note: if the line happens to pass through the origin, (0, 0) then both the x- and y-intercept are the same point: (0, 0). In that case you would need to find a second point by letting either x or y be some convenient value, say $x = 1$, and solve for the remaining coordinate of the second point.

Isolines

Consider a family of lines in standard form $Ax + By = C$, where A and B are fixed but C varies. For example:



$3x + 2y = 6$

$3x + 2y = 9$

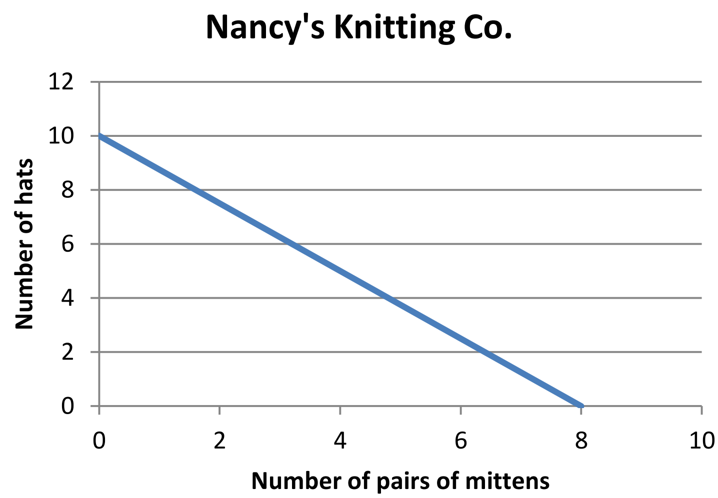
$3x + 2y = 12$

When the intercepts of these lines are found and the lines drawn, you will see that the result is a series of parallel lines. These are called **Isolines**, and you can think of them as a “family” of lines.

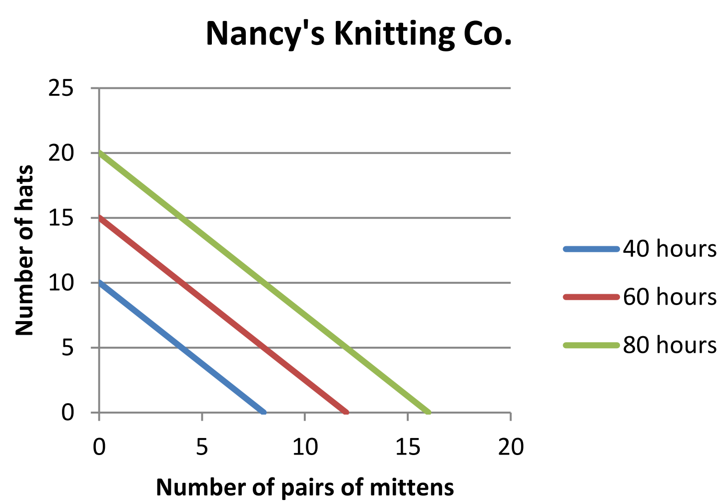
(By the way, if you like to memorize formulas, for any line in standard form $Ax + By = C$, the x-intercept is (0, C/B) and the y-intercept is (C/A, 0).)

Why are we interested in Isolines? Let’s look at an example:
Nancy’s Knitting company makes mittens and hats. It will take 5 hours of labor to make a pair of mittens and 4 hours of labor to make a hat. How many of each can be made if there is a total of 40 hours of

labor available? There are two unknowns, the number of mittens and the number of hats. Note that these two variables cannot be readily classified as an independent or dependent variable. They are related of course, the more time we put into knitting mittens, the less time is available for knitting hats, but one is not independent of the other. Let m = # mittens, h = # of hats, each pair of mittens takes 5 hours so 5m is the total hours needed to make m mittens. And 4h would be the total number of hours to make h hats. So our model: $5m + 4h = 40$ represents the relationship for how many hats and how many mittens can be made in 40 hours.



This equation has many possible solutions: 4 pairs of mittens and 5 hats, 10 hats and no mittens, 8 pairs of mittens and no hats, etc. When the line is graphed, each point on the line represents a possible combination for the number of hats and mittens that can be made with 40 hours of labor.



Taking this example a step further, what if Nancy hires a neighbor who is willing to knit 20 hours a week giving a total of 60 hours. The equation then becomes:

$5m + 4h = 60$

What if 80 hours of labor are available?

$5m + 4h = 80$

L hours of labor? $5m + 4h = L$

Each point, on each line shows a possible way to divide the available labor force between the two product lines for a given amount of labor.

We will leave this topic for the moment and come back to it in Module 4.

Systems of Equations

Systems of equations should be review material from previous algebra classes. A system of two equations in two variables such as:

$3x + 2y = 7$
$4x - y = 2$

Could be solved by different methods: the graphing method, the substitution method or "cancellation" method (aka elimination or combination). Depending on a specific system one method might be preferred over another but any one should work.

Using the cancellation method to solve:

$3x + 2y = 7 \quad\quad\quad\quad\quad\quad \rightarrow 3x + 2y = 7$
$4x - y = 2 \rightarrow 2(4x - y = 2) \rightarrow 8x - 2y = 4$

Adding the two equations together: $11x \quad = 11 \rightarrow x = 1$

Back substituting: $3(1) + 2y = 7 \rightarrow 2y = 4 \rightarrow y = 2$

The final solution to the system is then $(1, 2)$. Note that we have two variables, so we need two values for a solution. The solution to a system of equations should make each equation in the system true:

$$3(1) + 2(2) = 7$$
$$4(1) - (2) = 2$$

Standard Story Problem
An investment of $3000 is placed in stocks and bonds. The annual return on the stocks is 4.5%, while on the bonds it is 8%. The annual return from the stocks and bonds is $177. How much is invested in stocks and how much in bonds?

s = amount of the investment in stocks, $0.045s$ = amount of the return on the stocks
b = amount of the investment in bonds, $0.08b$ = amount of the return on the bonds

System of equations: $s + b = 3000$
$0.045s + 0.08b = 177$

Substitution: $s + b = 3000 \rightarrow s = 300 - b$

then: $0.045(3000 - b) + 0.08b = 177$

and: $135 - .045b + .08b = 177$

$0.035b = 42$ and $b = 1200$ and $s = 1800$

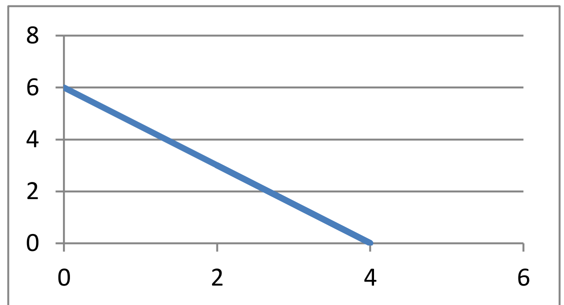
$1200 was invested in bonds and $1800 was invested in stocks.

Systems of Inequalities

Now that we know how to graph a linear function Ax + By = C by using the x and y intercepts, we will take this concept one step further and use it to graph **inequalities** such as Ax + By > C and Ax + By < C.

General process:

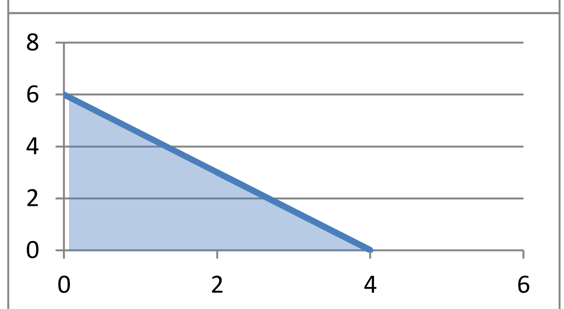
- Graph the line Ax + By = C.
- Pick a test point on one side of the line or the other (I like to use the point (0,0) whenever possible).
- Check whether the test point “satisfies” the inequality (does it make it true, or does it make it false?)
- Shade the side of the line where the points are true.



Example of a system of inequalities:

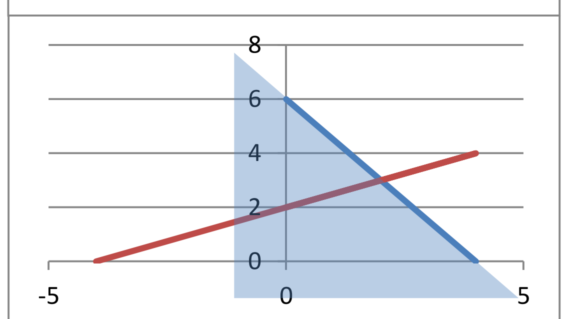
3x + 2y <= 12
x – 2y <=-4,
x >=0
y>=0

First we graph 3x + 2y = 12 by locating the x-intercept (4, 0) and the y-intercept (0, 6) and connecting the points.



Second, we pick a test point on one side of the line or other, I choose the point (0, 0): 3(0) + 2(0) <= 12 true or false? True, so shade the side of the line that includes the point (0,0)

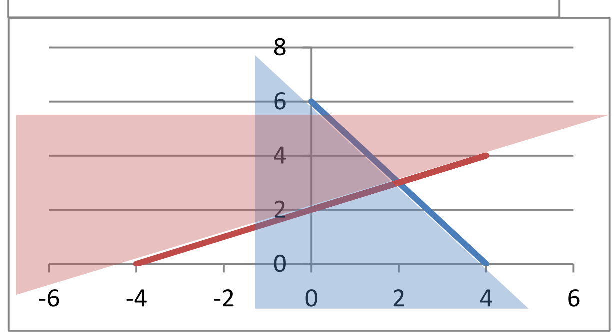
Do the same with x – 2y <= -4 on the same graph:



x-intercept (-4, 0) y-intercept: (0, 2)

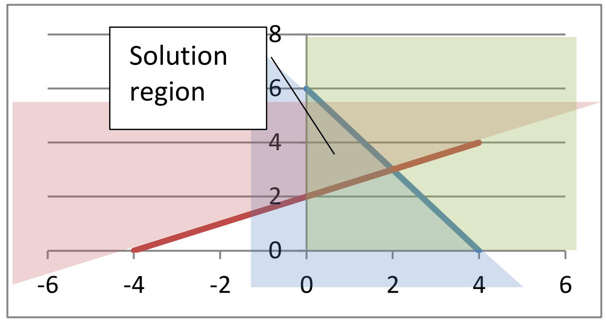
Using (0, 0) as a test point again, is 0 – 2(0) <= -4? False, so shade the side of the line that does NOT contain (0, 0).

Now add in x>=0 and y>=0:



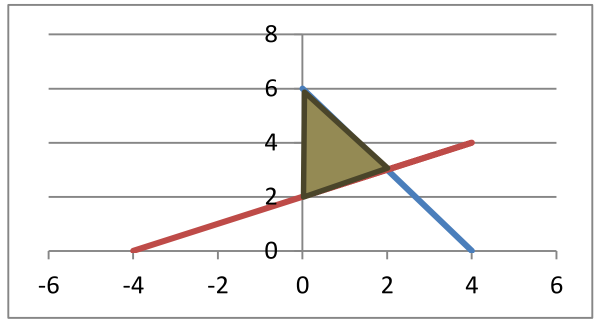
The line x = 0 is the y-axis (vertical axis) and x is greater than zero to the right of the y-axis.

The line y = 0 is the x-axis (horizontal axis) and y is greater than zero above the x-axis. These two constraints, x>=0 and y>=0, will keep our solution region in the first quadrant. Almost all of the problems we look at will have these two constraints as for many of our application problems negative values will not make sense.



Graphing these last two inequalities results in an enclosed region. It is a triangular are that is the overlapping area of each shaded region. Any point within this region is a solution to each of the inequalities in the system.

There are 3 corners to this region. Pretty easy to see that (0, 6) and (0,2) are corners. The third corner is (2, 3). How could we find that point? That point is the intersection of the lines 3x + 2y = 12 and x – 2y = -4. Solving the system of equations will give us the point (2, 3)



To graphically solve a **System of Inequalities:**

- Graph each line Ax + By = C
- Pick a point not on the line but clearly on one side or the other, and test that point in the inequality: Ax + By >C (or Ax + By < C) and determine whether it makes the inequality true or false. If it makes it true, shade the side of the line that the point is on. If the point makes the inequality false, then shade the opposite side of the line.
- Repeat for each inequality in the system.
- If x>=0 and y>=0 are part of your system, shade quadrant I.
- Identify what region is overlapped by all shaded areas. Determine the coordinates of every corner of the solution region. (Why? Those corner points will be important in the next Module.)

A **solution** to a System of Inequalities is the set of points that make every inequality in the system true. If there is not a region that is overlapped by all of the shaded areas, then the system does not have a solution.